B.Sc. Physics Part-I
Paper-I
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Harmonic Oscillator:

Why study harmonic oscillator –

It is not only exactly solvable in classical (as well as quantum theory) but it is a system of great physical relevance. Concepts are extended to related problems.

For example, any system having small fluctuation near a stable equilibrium configuration may be described by a single harmonic ~~oscillator~~ oscillator or by a collection of decoupled harmonic oscillator.

Example of a single harmonic oscillator is a mass m coupled to a spring of force constant k.

For small deformations x, spring will exert the force given by Hooke's law $F = -kx$,

k is its force constant.

and it produce a potential $V = \frac{1}{2} k x^2$.

The total energy for this system is $E = T + V$

or $$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2,$$

where $\omega = \sqrt{k/m}$.

ω is classical frequency of oscillation.

Mass-spring system is one example of harmonic oscillator.
Now consider a particle moving in a potential $U(x)$.
If particle is placed at one of its minima x_0, it will remain there in a state of stable static equilibrium.

~~as~~ We can write the dynamics of the particle as it fluctuates by small amounts near $x = x_0$.
The Potential $U(x)$ may be expanded about x_0 by using Taylor series:

$$U(x) = U(x_0) + \frac{dU}{dx}\Big|_{x=x_0}(x-x_0) + \frac{1}{2!}\frac{d^2U}{dx^2}\Big|_{x=x_0}(x-x_0)^2 + \cdots$$

$U(x_0)$ has no physical consequence and may be dropped
One can choose it as the arbitrary reference point for determining the potential.

$$\text{So} \quad U(x) = \frac{1}{2!}\frac{d^2U}{dx^2}\Big|_{x=x_0=0}(x-x_0)^2 + \frac{1}{3!}\frac{d^3U}{dx^3}\Big|_{x=x_0=0}(x-x_0)^3 + \cdots$$

For small oscillations, we can neglect higher order term and obtain.

$$U(x) = \frac{1}{2}\frac{d^2U}{dx^2}\Big|_{x=x_0=0}(x-x_0)^2$$

$$\text{or} \quad U(x) = \frac{x^2}{2}\frac{d^2U}{dx^2}$$

$\frac{d^2U}{dx^2} \longrightarrow$ identical to $m\omega^2 = k$. { see the example on page-01 }

Simple Harmonic motion:

A particle is said to perform a simple harmonic motion (SHM) if its acceleration is proportional to its displacement from its equilibrium position ~~and is alway~~ or any other point in its path and is always directed towards it.

The force F for SHM is given by $F = -kx$

where k is a positive constant.

Now using Newton's second law of motion, we can wrote

$$m\frac{d^2x}{dt^2} = -kx$$

or $$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

differential equation of motion of a simple harmonic motion.

Let $\frac{k}{m} = \omega^2$, we can write above equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} = -\omega^2 x$$

multiplying both sides of the equation by $2\frac{dx}{dt}$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -2\omega^2 x\frac{dx}{dt}$$

Or $$\int 2\frac{dx}{dt}\frac{d^2x}{dt^2}dt = -\int \omega^2 2x\frac{dx}{dt}dt$$ (Integrating w.r.t t)

$$\int \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 dt = -\int \omega^2 \frac{d}{dt}(x^2)dt$$

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + C$$

constant of Integration.

at maximum displacement, $\left.\frac{dx}{dt}\right|_{x=a} = 0$, we can write

$$0 = -\omega^2 a^2 + C$$

$$\Rightarrow C = \omega^2 a^2$$

Now $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + \omega^2 a^2 = \omega^2(a^2 - x^2)$

or $\frac{dx}{dt} = \omega\sqrt{a^2 - x^2}$, $\frac{dx}{dt} \rightarrow$ velocity of the particle at an instant t.

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega\, dt$$

integrating above eqn w.r.t. t

$$\sin^{-1}\frac{x}{a} = \omega t + \phi$$

$$\boxed{x = a\sin(\omega t + \phi)}$$

{displacement of the particle at an instant t}

$a \rightarrow$ amplitude

$\omega t + \phi \rightarrow$ total phase

$\phi \rightarrow$ phase constant

At. $x = 0 \rightarrow t = 0$, we have $\phi = 0$, and can write $\boxed{x = a\sin\omega t.}$

when $x = a$, at $t = 0$, we have $a = a\sin(\phi)$

$$\phi = \pi/2.$$

or $\boxed{x = a\sin(\omega t + \pi/2)}$

Time period - $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$ $\rightarrow$ independent of a, ϕ.

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